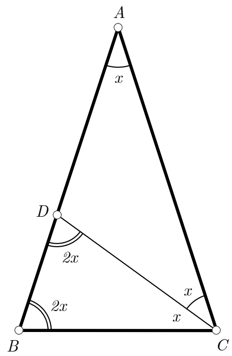
Problem 5134. Given isosceles triangle $\triangle ABC$ with cevian CD such that $\triangle CDA$ and $\triangle CDB$ are also isosceles, find the value of

$$\frac{AB}{CD} - \frac{CD}{AB}$$

Proposed by Kenneth Korbin, New York, NY

Solution by Ercole Suppa, Teramo, Italy



Denote $\angle BAC = x$. Since $\triangle CDA$ is isosceles we have $\angle DCA = x$. The exterior angle theorem yields $\angle BDC = 2x$ and this implies that $\angle ABC = 2x$.

Therefore, since $\triangle ABC$ is isosceles, we have $\angle BCA = 2x$. Thus

$$\angle A + \angle B + \angle C = 180^\circ \qquad \Leftrightarrow \qquad 5x = 180^\circ \qquad \Leftrightarrow \qquad x = 36^\circ$$

By using the sine law and bearing in mind the known formulas

$$\sin 36^\circ = \frac{1}{4}\sqrt{10 - 2\sqrt{5}} \qquad , \qquad \sin 108^\circ = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$$

we obtain

$$\begin{aligned} \frac{AB}{CD} - \frac{CD}{AB} &= \frac{\sin 108^\circ}{\sin 36^\circ} - \frac{\sin 36^\circ}{\sin 108^\circ} = \frac{\sin^2 108^\circ - \sin^2 36^\circ}{\sin 36^\circ \cdot \sin 108^\circ} = \\ &= \frac{\frac{5+\sqrt{5}}{8} - \frac{5-\sqrt{5}}{8}}{\frac{\sqrt{80}}{16}} = \frac{\frac{\sqrt{5}}{4}}{\frac{\sqrt{80}}{16}} = 1 \end{aligned}$$

This completes the solution. $\square$